Lecture I. Heat equation:

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Let $u(t,x)$ denote the temperature at point x at time t.

One dimensional heat equation is:

$$\partial_t u = k\,\partial_x^2 u$$

where $k>0$ is a constant (the thermal conductivity of the material)

That is, the change in heat at a specific point is proportional to the second derivative of the heat along the wire.

- Method of separation of variables

we write $u(t,x) = Z(t)\,y(x)$, a product of a function of t and a function of x.

we obtain,
$$\partial_t u(t,x) = Z'(t)\,y(x)$$
$$\partial_x^2 u(t,x) = Z(t)\,y''(x)$$

Thus. $Z'(t)\,y(x) = k\,Z(t)\,y''(x)$

assuming $Z(t)$, $y(x)$ are non-zero,

we have $\dfrac{Z'(t)}{Z(t)} = \dfrac{y''(x)}{y(x)} = \lambda$ $\quad k=1$

since the left-hand side is a constant with respect to x, and the right-hand side is a constant w.r.t t.

both sides must be constant.

$$\Rightarrow \begin{cases} Z'(t) - \lambda Z(t) = 0 \\ y''(x) - \lambda y(x) = 0 \end{cases}$$

$\lambda < 0$: physical solution

Let $\lambda = -w^2$. Then, $\begin{cases} z'(t) + w^2 z(t) = 0 & \text{①} \\ y''(x) + w^2 y(x) = 0 & \text{②} \end{cases}$

The general solution of ② takes the form

$$y(x) = a_w \cos(wx) + b_w \sin(wx)$$

The general solution of ① is

$$z(t) = c \exp(-w^2 t)$$

So we have $u(t.x) = z(t)\, y(x)$

$$= \exp(-w^2 t)\left(A_w \cos(wx) + B_w \sin(wx)\right)$$

observe that superposition (linear combination) is also a solution. Thus.

$$u(t.x) = c_0 + \sum_{w=0}^{\infty} \exp(-w^2 t)\left(A_w \cos(wx) + B_w \sin(wx)\right)$$

$$= c_0 + \sum_{w=0}^{\infty} \left(a_w \cos(wx) + b_w \sin(wx)\right)$$

solve the equation.

Note that it is a representation in the form of a Fourier series with coefficients depending on the time t

II. wave equation

method of separation of variables:

(eq) $\partial_t^2 \Psi(t.x) = \partial_x^2 \Psi(t.x)$

Let $\Psi(t.x) = \underset{\text{shape}}{A(x)} \cdot \underset{\text{time evolution}}{B(t)}$

$\Rightarrow \quad A(x)\, B''(t) = B(t)\, A''(x)$

$\Rightarrow \quad \dfrac{B''(t)}{B(t)} = \dfrac{A''(x)}{A(x)} = \text{constant} = -w^2$

we get two equations: $B''(t) = -w^2 B(t)$.

$$A''(x) = -\omega^2 A(x)$$

$$\Rightarrow \quad B(t) = \alpha \cos(\omega t) + \beta \sin(\omega t)$$

$$A(x) = a \cos(\omega x) + b \sin(\omega x)$$

Boundary condition: $A(0) = A(\pi) = 0 \Rightarrow a = 0$

$$\Rightarrow \quad \Psi_\omega(t,x) = b \sin(\omega x) \underbrace{\left[\alpha_\omega \cos(\omega t) + \beta_\omega \sin(\omega t) \right]}_{A_\omega(t)}$$

By superposition, formally,

$$\Psi(t,x) = \sum_{\omega \geq 0}^{\infty} A_\omega(t) \sin(\omega x). \quad \text{solves (eq)}$$

where $A_\omega(t) = \underline{\alpha_\omega} \cos(\omega t) + \underline{\beta_\omega} \sin(\omega t)$

By initial conditions

Consider initial conditions: $\Psi(0,x) = f(x)$

$$\partial_t \Psi(0,x) = g(x)$$

we require that

$$\sum_{\omega \geq 0}^{\infty} \alpha_\omega \sin(\omega x) = f(x)$$

$$\sum_{\omega \geq 0}^{\infty} \omega \beta_\omega \sin(\omega x) = g(x)$$

This raises the question:

Given f on $[0,\pi]$, $f(0) = f(\pi) = 0$. can we find α_m such that $f(x) = \sum_{\geq 0}^{\infty} \alpha_\omega \sin(\omega x)$?

$\boxed{\text{Theorem 1:}}$ Let $f \in L^2(\mathbb{T})$. Then $\lim_{N\to\infty} \| S_N f - f \|_{L^2(\mathbb{T})} = 0$.

<u>Remark</u>: $S_N f(x) = \sum_{|k|\le N} \hat{f}(k) e^{2\pi i k x}$, $S_N g(x) = \sum_{|k|\le N} \hat{g}(k) e^{2\pi i k x}$

$$\int_0^1 S_N f(x) \overline{S_N g(x)}\, dx$$

$$= \int_0^1 \sum_{|k|\le N} \hat{f}(k) e^{2\pi i k x} \sum_{|m|\le N} \overline{\hat{g}}(m) e^{-2\pi i m x}\, dx$$

$$= \sum_{|k|,|m|\le N} \hat{f}(k) \overline{\hat{g}(m)} \underbrace{\int_0^1 e^{2\pi i (k-m) x}\, dx}_{\delta_{km}}$$

$$= \sum_{|k|\le N} \hat{f}(k) \overline{\hat{g}(k)}$$

In particular, $\| S_N f \|^2_{L^2(\mathbb{T})} = \sum_{|k|\le N} |\hat{f}(k)|^2$

Pf: 1° First we observe that $e_k(x) = e^{2\pi i k x}$ form an orthonormal basis basis of $L^2(\mathbb{T})$

2°. $S_N f = \sum_{|k|\le N} (f, e_k) e_k$

therefore. $(f - S_N f, e_j) = 0 \quad \forall |j| \le N$

Then the orthonormal property of the family $\{e_k\}$ implies that $f - S_N f$ is orthonormal to e_j for all $|j| \le N$. i.e.

$$(f - S_N f) \perp \operatorname{span}\{e_j : |j| \le N\}$$

In particular,

$$\|f\|^2_{L^2} = \|f - S_N f\|^2_{L^2} + \|S_N f\|^2_{L^2}$$

$$\ge \|S_N f\|^2_{L^2}$$

$$= \sum_{|k|\le N} |(f, e_k)|^2$$

$\Rightarrow S_N f \xrightarrow[N\to\infty]{} \sum_{k\in\mathbb{Z}} (f, e_k) e_k$ in L^2

$\Rightarrow \lim_{N\to\infty} \| S_N f - f \|_{L^2} = 0$ ☒

Pf of " $\widehat{f * g}(\xi) = \hat{f}\,\hat{g}$ "

$$\widehat{f * g}(\xi) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(f * g)(x)\, e^{-ix\xi}\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\, g(x-y)\, dy \;\; e^{-ix\xi}\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\, e^{-i\xi y}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} g(x-y)\, e^{-i\xi(x-y)}\, dx\right) dy$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\, e^{-i\xi y}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\, e^{-i\xi x}\, dx\right) dy$$

$$= \hat{f}(\xi)\;\hat{g}(\xi)$$